ON THE ACCURACY OF TRIGONOMETRIC INTERPOLATION*

BY

DUNHAM JACKSON

Let $f(x)$ be a function of x, with the period 2π, which takes on known values, determined by physical measurement or otherwise, at $2n+1$ points equally spaced throughout a period. It is well known that a finite trigonometric sum, of the nth order at most, which coincides with $f(x)$ at the points in question, is given by formulæ analogous to those which characterize the Fourier's development of $f(x)$, the integrals of the latter development being replaced by finite sums involving the known quantities. It has been found by FABER that in the case of continuous functions the ordinary sufficient conditions, in the way of restrictions on $f(x)$, for the convergence of the Fourier's series are sufficient also to insure the convergence of the interpolating function to the value $f(x)$ at all points, as the number n is indefinitely increased; although in cases not covered by these explicit conditions one development may converge while the other does not.† In the present note it is shown that a method by which the author has studied the rapidity of convergence of Fourier's series is adapted to the treatment of the corresponding problem in interpolation,‡ and yields similar results. In fact, these results are obtained by a simple combination of materials already at hand. It is further pointed out how a finite number of observed values may be used to define a formula of approximation which converges more rapidly in certain cases than the ordinary interpolation formula.

I. Ordinary Interpolation.

Let $f(x)$ be a function of x having the period 2π. Let $x_1, x_2, \cdots, x_{2n+1}$ be a set of values of x such that

$$x_{i+1} - x_i = \frac{2\pi}{2n+1} \qquad (i = 1, 2, \cdots, 2n).$$

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† FABER, *Mathematische Annalen*, vol. 69 (1910), pp. 372–443; see pp. 417–443. At the close of the article, FABER considers also certain classes of discontinuous functions. The sufficiency of the condition of limited variation had previously been deduced by DE LA VALLÉE POUSSIN, *Bulletins de l'Académie royale de Belgique*, Classe des Sciences, 1908, pp. 319–403.

‡ DE LA VALLÉE POUSSIN (loc. cit., p. 389) obtains the result that if $f(x)$ possesses a derivative having limited variation, the error of the interpolation formula nowhere exceeds a constant multiple of $1/n$. This theorem is not among those obtained in the present paper.

If the coefficients in the trigonometric sum

$$S_n(x) = \frac{a_0}{2} + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx$$

are determined by the formulæ

$$a_h = \frac{2}{2n+1} \sum_{i=1}^{2n+1} f(x_i) \cos hx_i, \qquad b_h = \frac{2}{2n+1} \sum_{i=1}^{2n+1} f(x_i) \sin hx_i \qquad (h = 0, 1, \cdots, n),$$

the sum can be expressed in the form*

$$S_n(x) = \frac{1}{2n+1} \sum_{i=1}^{2n+1} f(x_i) \frac{\sin[(n+\frac{1}{2})(x_i - x)]}{\sin \frac{1}{2}(x_i - x)},$$

by means of a familiar identity, and from this expression it appears that $S_n(x_j) = f(x_j)$ at each of the $2n + 1$ points x_j, all the terms but one under the sign of summation being zero. It is the only trigonometric sum of order n or lower which has this property, since two such sums which are equal for more than $2n$ distinct values (mod 2π) of the argument must be identical.

It was proved by FABER† that if $f(x)$ never exceeds 1 in absolute value, then the absolute value of $S_n(x)$ can not exceed a certain constant multiple of $\log n$. The proof, which may be given here for the sake of completeness, is as follows. With the hypothesis made concerning $f(x)$,

$$|S_n(x)| \leqq \frac{1}{2n+1} \sum_{i=1}^{2n+1} \left| \frac{\sin[(n+\frac{1}{2})(x_i - x)]}{\sin \frac{1}{2}(x_i - x)} \right|.$$

Set $\frac{1}{2}(x_i - x) = u_i$; then

$$|S_n(x)| \leqq \frac{1}{2n+1} \sum_{i=1}^{2n+1} \left| \frac{\sin(2n+1)u_i}{\sin u_i} \right|.$$

In view of the periodicity of the quotient, it may be assumed without loss of generality that all the variables u_i lie in the interval from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$. Let v_0 be the smallest of the numbers $|u_i|$ (or one of the two smallest, if two are equal), v_1 the second, and so on. Then we have, on rearranging the sum,

$$|S_n(x)| \leqq \sum_{i=0}^{2n} \left| \frac{\sin(2n+1)v_i}{(2n+1)\sin v_i} \right|,$$

where, as is readily seen,

$$\frac{i\pi}{2(2n+1)} \leqq v_i \leqq \frac{(i+1)\pi}{2(2n+1)}.$$

Now

* Throughout this paper, an expression which, regarded as a quotient, becomes meaningless for certain values of the variable by taking on the form $0/0$, will be understood to be defined for those values so as to be a continuous function.

† Loc. cit., pp. 419–420.

(1) $$\left|\frac{\sin(2n+1)\,v_0}{(2n+1)\sin v_0}\right| = \left|\frac{\sin(2n+1)\,v_0}{(2n+1)\,v_0}\cdot\frac{v_0}{\sin v_0}\right| \overline{\overline{<}}\, 1,$$

since $(\sin v)/v$ is a monotonically decreasing function in the interval $(0, \pi)$ which includes both the points v_0 and $(2n+1)\,v_0$. Similarly, the absolute value of the term in v_1 does not exceed 1. For $i \overline{\overline{>}}\, 2$,

$$\left|\frac{\sin(2n+1)\,v_i}{(2n+1)\sin v_i}\right| \overline{\overline{<}}\, \frac{1}{(2n+1)\sin v_i};$$

since $\sin v \overline{\overline{>}}\, 2v/\pi$ for $0 \overline{\overline{<}}\, v \overline{\overline{<}}\, \frac{1}{2}\pi$, we may write

(2) $$(2n+1)\sin v_i \overline{\overline{>}}\, (2n+1)\sin\frac{i\pi}{2(2n+1)} \overline{\overline{>}}\, i,$$

and hence

$$\sum_{i=2}^{2n}\left|\frac{\sin(2n+1)\,v_i}{(2n+1)\sin v_i}\right| \overline{\overline{<}}\, \sum_{i=2}^{2n}\frac{1}{i} < \log 2n.$$

Consequently

$$|S_n(x)| < 2 + \log 2 + \log n < 5 \log n \qquad (n \geqq 2).$$

It would be easy to replace the constant 5 by a smaller value, but we shall not make any effort to do so here.

From the inequality just obtained, it follows immediately that if $|f(x)| \overline{\overline{<}}\, \epsilon$, then $|S_n(x)| \overline{\overline{<}}\, 5\epsilon \log n$, and $|f(x) - S_n(x)| \overline{\overline{<}}\, \epsilon + 5\epsilon \log n \overline{\overline{<}}\, 7\epsilon \log n$, $n \overline{\overline{>}}\, 2$.

Now let the restriction on $f(x)$ be replaced by the broader hypothesis that there exists a trigonometric sum $T_n(x)$, of the nth order at most, such that for all values of x we have the inequality

$$|f(x) - T_n(x)| \overline{\overline{<}}\, \epsilon.$$

The interpolating function $S_n(x)$ that corresponds to $f(x)$ may be obtained by adding those for $T_n(x)$ and for $f(x) - T_n(x)$. The former is identical with $T_n(x)$, being equal to it at $2n+1$ points, and so an error is committed only in approximating to the difference $f(x) - T_n(x)$, which does not exceed ϵ. It follows that the error can never be greater than $7\epsilon \log n$ if $n \geqq 2$, and we have, to sum up,*

THEOREM I. *If $f(x)$ can be approximately represented by a finite trigonometric sum of the nth order at most, $n \geqq 2$, with an error never exceeding ϵ, then $f(x)$ is represented by its interpolating function $S_n(x)$ with an error not exceeding $7\epsilon \log n$.*

This can be immediately combined with theorems of the author concerning the accuracy of approximation by trigonometric sums† to give a variety of more explicit results. To mention only two:

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* Cf. LEBESGUE, *Sur les intégrales singulières*, Annales de la Faculté de Toulouse, ser. 3, vol. 1 (1910), pp. 25–117, for the corresponding result in the case of Fourier's series (see pp. 116–117).

† These Transactions, vol. 13 (1912), pp. 491–515. This article will be referred to hereafter as A. Also, these Transactions, pp. 343–364 of the present volume.

If $\lim_{\delta=0}[f(x+\delta)-f(x)]\log|\delta|=0$, uniformly for all values of x, then $S_n(x)$ uniformly approaches $f(x)$ as its limit when n is indefinitely increased.*

If $f(x)$ everywhere satisfies the Lipschitz condition†

$$|f(x'')-f(x')| \leqq \lambda\,|x''-x'|,$$

then, for all values of x, and for $n \geqq 2$,

$$|f(x)-S_n(x)| \leqq \frac{21\lambda\log n}{n}.$$

The preceding work also makes it easy to estimate the possible effect of errors‡ in the determination of the quantities $f(x_i)$. Let $S_n(x)$ be the interpolating function defined by the true values $f(x_i)$, and $\bar{S}_n(x)$ the function defined by a set of observed values $\bar{f}(x_i)$. The difference between these two functions is the corresponding function defined by the differences $f(x_i)-\bar{f}(x_i)$, and, if the greatest of the latter does not exceed ϵ, the maximum of $|S_n(x)-\bar{S}_n(x)|$ can not exceed $5\epsilon\log n$. It would be possible to replace the coefficient 5 by a smaller one, as has been said before, but it is interesting to remark that it would not be possible to dispense with the factor $\log n$, or even to replace it by one which becomes infinite less rapidly with n, since, as Faber shows, the maximum of $|S_n(x)|$ for functions $f(x)$ not exceeding 1 in absolute value actually is of the order§ of $\log n$.

II. A Modified Formula.

For comparison with $S_n(x)$, let us define a function $\Sigma_m(x)$ as follows:

$$\Sigma_m(x) = H_m\sum_{i=1}^{2m} f(x_i)\left[\frac{\sin\frac{1}{2}m(x_i-x)}{m\sin\frac{1}{2}(x_i-x)}\right]^4, \tag{3}$$

where

$$\frac{1}{H_m}=\sum_{i=1}^{2m}\left[\frac{\sin\frac{1}{2}m(x_i-x)}{m\sin\frac{1}{2}(x_i-x)}\right]^4.$$

The points x_i are assumed to be such that‖

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* This was deduced by Faber (loc. cit., p. 422); the particular theorem used here concerning approximation by trigonometric sums was proved by Lebesgue, loc. cit., pp. 115–116.

† See A, Theorems I, VI.

‡ The fact that, in the case of polynomial interpolation with equidistant ordinates, errors of observation may produce a disproportionately large effect, was pointed out by de la Vallée Poussin, loc. cit., pp. 321–322; see also p. 346 of the same article.

§ Faber, loc. cit., p. 424. If $f(x_i)=(-1)^{i-1}$, and $x_1-x=\pi/(2n+1)$, then

$$f(x_i)\sin[(n+\tfrac{1}{2})(x_i-x)]=+1$$

for all values of i, and it is readily seen that $S_n(x)$ is of the order of $\log n$.

‖ It would be possible, without essential change, to use instead an odd number, $2m-1$, of equally spaced points.

$$x_{i+1} - x_i = \frac{\pi}{m} \qquad (i = 1, 2, \cdots, 2m-1).$$

Let us first notice that the apparent dependence of H_m on the variable x is illusory; H_m really depends only on m. This may be seen as follows. The fourth power under the sign of summation can be expressed in the form*

$$\begin{aligned} A_{m0} + A_{m1} \cos(x_i - x) + A_{m2} \cos 2(x_i - x) + \cdots \\ + A_{m,\,2m-2} \cos[(2m-2)(x_i - x)] \\ = \sum_{h=0}^{2m-2} A_{mh} (\cos hx_i \cos hx + \sin hx_i \sin hx), \end{aligned}$$

the coefficients A_{mh} being constants. It is a well-known property of the trigonometric functions that if $y_1, \cdots, y_q$ are q points disposed at successive intervals of $2\pi / q$, and p is not divisible by q, then†

$$\sum_{i=1}^{q} \cos py_i = \sum_{i=1}^{q} \sin py_i = 0.$$

For the application that we wish to make, q is $2m$, and p is one of the numbers $0, 1, \cdots, 2m-2$, and so the sum defining $1 / H_m$ reduces to $2mA_{m0}$.

From this fact and the expression used above for the fourth power involved, it appears that $\Sigma_m(x)$ is a trigonometric sum in x, of order not higher than $2(m-1)$. It is a formula of interpolation, in the sense that it is completely determined by a finite number of values of the function represented, though it does not, like $S_n(x)$, become equal to $f(x)$ at the points x_i, in spite of the fact that it is of higher order than $S_n(x)$ in proportion to the number of points used. We shall proceed to establish the property which constitutes its chief claim to attention: If $f(x)$ satisfies a Lipschitz condition, it is represented by $\Sigma_m(x)$ with an error which approaches zero at least as fast as $1 / n$ in order of magnitude, if we denote by n the maximum order $2m-2$ of the trigonometric sum.

Suppose that $f(x)$ satisfies everywhere the Lipschitz condition

$$|f(x'') - f(x')| \overline{\gtrless} \lambda |x'' - x'|.$$

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* See A, p. 493. The absence of sine-terms follows from the fact that the expression is an even function of $x_i - x$.

† Cf. e. g. Bôcher, *Introduction to the theory of Fourier's series*, Annals of Mathematics, ser. 2, vol. 7 (1906), pp. 81–152; p. 135. For an analytic proof, a starting point is the fact that when $p = 1$ and one of the y_i's is zero, $\Sigma \cos y_i$ and $\Sigma \sin y_i$ are the real and imaginary parts respectively of the sum of the qth roots of unity, which is zero if $q > 1$. Then the cases may be considered successively that p is any number relatively prime to q, and that p and q have a greatest common divisor greater than 1 but less than q, still with the assumption that one of the y_i's is zero. The last generalization may be effected by letting $y_i = y_1 + y_i'$, where y_1 is arbitrary.

We may write

$$(4)\qquad \Sigma_m(x) - f(x) = H_m \sum_{i=1}^{2m} [f(x_i) - f(x)] \left[\frac{\sin \frac{1}{2}m(x_i - x)}{m \sin \frac{1}{2}(x_i - x)}\right]^4,$$

or, setting $\frac{1}{2}(x_i - x) = u_i$,

$$(5)\qquad \begin{aligned} |\Sigma_m(x) - f(x)| &= \left| H_m \sum_{i=1}^{2m} [f(x + 2u_i) - f(x)] \left[\frac{\sin mu_i}{m \sin u_i}\right]^4 \right| \\ &\overline{\underset{<}{}} \; 2\lambda H_m \sum_{i=1}^{2m} |u_i| \left[\frac{\sin mu_i}{m \sin u_i}\right]^4. \end{aligned}$$

Because of the periodicity of the functions involved in the first line of this last relation, we may assume without loss of generality that all the variables u_i there, and so also in the following line, are in the interval from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$.

The work of determining approximately the magnitude of H_m has essentially been performed in the article A. For, if we recall the notation J_m of that article,* we have

$$J_m = m \int_0^{\pi/2} \left[\frac{\sin mu}{m \sin u}\right]^4 du = m \int_0^{\pi/2} A_{m0}\, du = \frac{m\pi}{2} A_{m0} = \frac{\pi}{4} \cdot \frac{1}{H_m}.$$

As it was shown that $\frac{1}{2}\pi \geqq J_m > \frac{1}{3}\pi$, for all values of m, it follows that $\frac{1}{2} \leqq H_m < \frac{3}{4}$.

In the sum which still remains to be evaluated, let the smallest of the quantities $|u_i|$ (or, if this is indeterminate, one of the two smallest) be set equal to v_0, the next to v_1, and so on. Then

$$\sum_{i=1}^{2m} |u_i| \left[\frac{\sin mu_i}{m \sin u_i}\right]^4 = \sum_{i=0}^{2m-1} v_i \left[\frac{\sin mv_i}{m \sin v_i}\right]^4,$$

where

$$\frac{i\pi}{4m} \leqq v_i \leqq \frac{(i+1)\pi}{4m}.$$

Adapting the relation (1) to the present case, we see that

$$\left[\frac{\sin mv_0}{m \sin v_0}\right]^4 \leqq 1, \qquad \left[\frac{\sin mv_1}{m \sin v_1}\right]^4 < 1.$$

For $i \geqq 2$, by reasoning similar to that employed in connection with (2),

$$m \sin v_i \geqq m \sin \frac{i\pi}{4m} \geqq \frac{i}{2},$$

and

$$\left[\frac{\sin mv_i}{m \sin v_i}\right]^4 \leqq \left(\frac{2}{i}\right)^4.$$

Hence

* A, pp. 503–507.

$$\sum_{i=0}^{2m-1} v_i \left[\frac{\sin mv_i}{m \sin v_i}\right]^4 \overline{\overline{<}} v_0 + v_1 + \sum_{i=2}^{2m-1} v_i \left(\frac{2}{i}\right)^4.$$

Now $v_0 + v_1 = \pi / (2m)$, as is seen by reference to the definition of the symbols v_i; and

$$\sum_{i=2}^{2m-1} v_i \left(\frac{2}{i}\right)^4 \overline{\overline{<}} \sum_{i=2}^{2m-1} \frac{(i+1)\pi}{4m}\left(\frac{2}{i}\right)^4 = \frac{4\pi}{m}\sum_{i=2}^{2m-1}\frac{i+1}{i^4} < \frac{4\pi}{m}\left(\int_1^\infty \frac{du}{u^3} + \int_1^\infty \frac{du}{u^4}\right)$$

$$= \frac{4\pi}{m}\left(\frac{1}{2}+\frac{1}{3}\right) = \frac{10\pi}{3m}.$$

Consequently

$$\sum_{i=0}^{2m-1} v_i \left[\frac{\sin mv_i}{m \sin v_i}\right]^4 < \frac{\pi}{m}\left(\frac{1}{2}+\frac{10}{3}\right) = \frac{23\pi}{6m},$$

and so, by (5) and the inequality for H_m,

$$| \Sigma_m(x) - f(x) | \overline{\overline{<}} \frac{23\pi}{4} \cdot \frac{\lambda}{m}.$$

If we denote by n, as was suggested before, the maximum order of $\Sigma_m(x)$, which is less than $2m$, we have finally:

THEOREM II. *If the function $f(x)$, of period 2π, everywhere satisfies the Lipschitz condition $|f(x'') - f(x')| \overline{\overline{<}} \lambda | x'' - x' |$, it is approximately represented by the finite trigonometric sum $\Sigma_m(x)$, of order not higher than n, with an error not exceeding $37\lambda / n$.*

The coefficient 37 is not particularly significant; no attempt has been made to reduce it to as small a value as possible. The main point is that we have found here an upper limit for the error which is of higher order in its approach to zero than that which we had for the ordinary interpolation formula. We have not proved, to be sure, that the order of the limit of error which we found in the earlier case is the best that can be obtained, but it is not difficult to show, by the method applied by LEBESGUE* to the corresponding problem in the case of Fourier's series, that the error of the approximating function $S_n(x)$ for a function $f(x)$ satisfying a Lipschitz condition can actually exceed a constant multiple of $(\log n) / n$ for infinitely many values of n. The expression $\Sigma_m(x)$ accordingly is really superior to the other in the order of its approximation to certain functions of the class considered.

It may be observed further that errors in the quantities $f(x_i)$, none of which exceeds ϵ in absolute value, can not affect $\Sigma_m(x)$ at any point by more than ϵ, as compared with a quantity of the order of $\epsilon \log n$ in the case of $S_n(x)$, and that $\Sigma_m(x)$ converges uniformly to the value $f(x)$, as m is indefinitely

* LEBESGUE, Bulletin de la Soc. Math. de France, vol. 38 (1910), pp. 184–210; pp. 203–206.

increased, for every continuous function $f(x)$. The former assertion is coextensive with the statement that $|\Sigma_m(x)|$ is never greater than 1 if $|f(x)|$ never exceeds 1, a fact which is obvious from the definition of $\Sigma_m(x)$.

To prove the convergence property suppose that ϵ is an arbitrary positive quantity. Let δ be a positive number such that $|f(x'') - f(x')| \leqq \frac{1}{2}\epsilon$ whenever $|x'' - x'| < \delta$; the existence of δ is a consequence of the hypothesis that $f(x)$ is everywhere continuous. Let x have any particular value, and divide the sum on the right hand side of (4) into two parts, one containing the terms* for which $|x_i - x| \leqq \delta$, and the other, the remaining terms. The first part does not exceed $\frac{1}{2}\epsilon$ in absolute value, for any value of m, in consequence of the definition of δ. In the second part of the sum, $|f(x_i) - f(x)|$ does not exceed twice the maximum of $|f(x)|$, and $\sin\frac{1}{2}(x_i - x) \geqq \sin\frac{1}{2}\delta$; consequently each term of the second part is inferior in absolute value to $1/m^4$ multiplied by a quantity independent of m and x, and the whole second part does not exceed a constant multiple of $1/m^3$. As soon as m is greater than a suitable quantity, independent of x, the second part too becomes smaller than $\frac{1}{2}\epsilon$, and the uniform convergence is established.

The properties of $\Sigma_m(x)$ last developed, while interesting in themselves, are less characteristic than the one on which the principal stress has been laid; they are shared, for example, by the simpler expression which is obtained if the fourth power in the definition of $\Sigma_m(x)$ is replaced by the corresponding square.

To apply the formula $\Sigma_m(x)$ to the problem of polynomial interpolation, the given ordinates, to be sure, being unequally spaced, suppose that $g(y)$ is a function of y defined in the interval $-1 \leqq y \leqq 1$. Suppose that the values of $g(y)$ are known, either at the m points

$$y_i = \cos\frac{(2i-1)\pi}{2m} \qquad (i = 1, 2, \cdots, m),$$

or at the $m + 1$ points

$$y_i = \cos\frac{i\pi}{m} \qquad (i = 0, 1, 2, \cdots, m).$$

If we set $y = \cos x$, $g(y) = g(\cos x) = f(x)$, we have the values of $f(x)$ given for a set of values of x symmetrically situated with respect to the origin, and equally spaced at intervals of π/m. Let those which lie in the interval $-\pi < x \leqq \pi$ be used as the points x_i for defining a function $\Sigma_m(x)$ corresponding to our present function $f(x)$. If we replace x by $-x$ in this expression, and at the same time replace x_i by $-x_i$, which amounts only to

* We suppose the numbers x_i replaced, if necessary, by values congruent to them (mod 2π), so that none of the differences $|x_i - x|$ is greater than π.

a rearrangement of the terms in the sum (3), because of the symmetrical arrangement of the points x_i, we see that $\Sigma_m(x)$ remains unchanged. Being an even function, it is a polynomial in $\cos x$, of degree $2(m-1)$ at most. We have thus a means of approximating to $g(y)$ by a polynomial in y. The properties of the trigonometric interpolation formula developed above can be immediately interpreted with reference to the present case.*

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.

* Cf. A, p. 495.